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The multisoliton solutions for the mKPI equation with self-consistent sources

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Abstract

The modified Kadomtsev–Petviashvili I equation with self-consistent sources (mKPIESCS) is derived through the linear problem of the modified Kadomtsev–Petviashvili I (mKPI) system. The bilinear form of the mKPIESCS is given and the *N*-soliton solutions are obtained through the Hirota method and the Wronskian technique, respectively. The coincidence of these solutions is shown by direct computation.

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1. Introduction

The soliton equations with self-consistent sources (SESCS) have received considerable attention in recent years. The reason may be that these equations are important models in many physical fields, such as hydrodynamics, soliton-state physics, plasma physics, etc [1–8]. Until now, there have been many ways to solve the SESCS. In [9–12] Zeng *et al* developed a simple treatment of the singularity in the evolution of eigenfunctions obtained from the explicit soliton solutions of some SESCS such as the KdV, modified KdV, nonlinear Schrödinger, AKNS and Kaup-Newell hierarchies with self-consistent sources, which were solved through the inverse scattering method. Recently, generalized binary Darboux transformations for some SESCS have been constructed and can be used to obtain *N*-soliton, positon and negaton solutions [13–16]. Also, some SESCS can be solved by the Hirota method and the Wronskian technique [17–20].

One of the purposes of this paper is to derive the hierarchy of the modified Kadomtsev–Petviashvili I equation with self-consistent sources (mKPIESCS) in the way which is directly based on the eigenfunctions of a recursion operator. This method, slightly different from that using constraint flows, is easy to obtain the Lax representations of the hierarchy. We have found that some other hierarchies of the SESCS, such as the KP equation with self-consistent sources [20], can also be obtained in this way. On the other hand, we also hope to find the multi-soliton solutions of the mKPIESCS through the Hirota method [21] and the Wronskian technique [22–24]. These two direct methods both depend on the bilinear forms of the evolution equations. The Hirota method provides a remarkably simpler technique for obtaining the N-soliton solutions in the form of an Nth-order polynomial in N exponentials. The Wronskian technique provides an alternative formulation of the N-soliton solutions, in terms of some function of the Wronski determinant of N functions, which allows verification of the solutions by direct substitution because differentiation of a Wronskian is easy and its derivatives take similar compact forms. We first present a set of dependent variable transformations to write out the bilinear form of the mKPIESCS by which we can derive one-, two-, even three-soliton solutions successively through the standard Hirota's approach. These results can help us to conjecture a general formula which denotes N-soliton solution but is only conjectured and not verified. Next, we can construct a Wronskian and try to verify it to satisfy the related bilinear equations. Since there is a nonlinear term (led by the concerned source) in the time evolution, we have to develop some novel determinantal identities and employ some special treatments which are different from the known standard Wronskian technique [22–24] so that we can finish the Wronskian verifications. Finally, we present a process to show that the solutions of the bilinear equations obtained through the above two direct methods are the same for recovering the solutions of mKPIESCS from the original dependent variable transformations. In other words, these two kinds of solutions are uniform. To our knowledge, it is the first time to obtain the mKPIESCS and solve it by the Hirota method and the Wronskian technique.

We arrange the paper as follows. We first derive the hierarchy of the mKPIESCS in section 2. Then we solve the mKPIESCS by means of Hirota method and Wronskian technique in sections 3 and 4, respectively. At last, in section 5 we show the uniformity of the results in sections 3 and 4.

2. The mKPI equation with self-consistent sources

Let us consider the spectral problem and its adjoint associated with the mKPI equation

$$\Phi_y = \Phi_{xx} + 2u\Phi_x, \tag{2.1}$$

$$\Psi_y = -\Psi_{xx} + 2u\Psi_x. \tag{2.2}$$

Suppose that the time evolution of the eigenfunction Φ is given by

$$\Phi_t = A\Phi, \tag{2.3}$$

where A is an operator function of ∂ and ∂^{-1} ($\partial = \frac{\partial}{\partial x}$ and $\partial^{-1}\partial = \partial\partial^{-1} = 1$). The compatibility of (2.1) and (2.3) requires that A satisfies

$$2u_t\partial - A_y + [\partial^2 + 2u\partial, A] = 0, \tag{2.4}$$

or

$$2u_t\partial = A_y - A_{xx} - 2A_x\partial - 2uA_x - 2[u, A]\partial. \tag{2.5}$$

Now we take

$$A = a_0\partial^3 + a_1\partial^2 + a_2\partial + \alpha(\Phi\Psi - \Phi\partial^{-1}\Psi_x), \tag{2.6}$$

where a_j ($j = 0, 1, 2$) are undetermined functions of u and its derivatives, and α is an arbitrary constant. Substituting (2.6) into (2.5) and equating coefficients powers of ∂, we obtain

$$2u_t = a_{2,y} - a_{2,xx} - 2ua_{2,x} + 2a_0u_{xxx} + 2a_1u_{xx} + 2a_2u_x - 2\alpha(\Phi\Psi)_x, \tag{2.7}$$

$$a_{1,y} - a_{1,xx} - 2a_{2,x} - 2ua_{1,x} + 6a_0u_{xx} + 4a_1u_x = 0, \tag{2.8}$$

$$a_{0,y} - a_{0,xx} - 2a_{1,x} - 2ua_{0,x} + 6a_0u_x = 0, \tag{2.9}$$

$$a_{0,x} = 0. \tag{2.10}$$

From (2.8)–(2.10), we work out that

$$a_0 = -4, \quad a_1 = -12u, \quad a_2 = -6\partial^{-1}u_y - 6u_x - 6u^2. \tag{2.11}$$

Substituting (2.11) into (2.7) and setting $\alpha = -1$, we obtain

$$u_t + u_{xxx} + 3\partial^{-1}u_{yy} - 6u^2u_x + 6(\partial^{-1}u_y)u_x - (\Phi\Psi)_x = 0. \tag{2.12}$$

This equation together with spectral problems (2.1) and (2.2) constitutes the mKPI equation with a self-consistent source. If taking $\alpha = 0$, we can derive the mKPI equation [25]

$$u_t + u_{xxx} + 3\partial^{-1}u_{yy} - 6u^2u_x + 6(\partial^{-1}u_y)u_x = 0. \tag{2.13}$$

In a similar way, the mKPI equation with N self-consistent sources can be defined , which is expressed as

$$u_t + u_{xxx} + 3\partial^{-1}u_{yy} - 6u^2u_x + 6(\partial^{-1}u_y)u_x - \sum_{j=1}^{N}(\Phi_j\Psi_j)_x = 0, \tag{2.14}$$

$$\Phi_{j,y} = \Phi_{j,xx} + 2u\Phi_{j,x}, \tag{2.15}$$

$$\Psi_{j,y} = -\Psi_{j,xx} + 2u\Psi_{j,x}, \tag{2.16}$$

while the operator A becomes

$$A = -4\partial^3 - 12u\partial^2 - (6\partial^{-1}u_y + 6u_x + 6u^2)\partial - \sum_{j=1}^{N}(\Phi_j\Psi_j - \Phi_j\partial^{-1}\Psi_{j,x}). \tag{2.17}$$

3. Solving the mKPIESCS by Hirota method

Now, we solve the mKPIESCS by the Hirota method. Through the dependent variable transformations

$$u = \left(\ln\frac{g}{f}\right)_x, \qquad \Phi_j = \frac{h_j}{g}, \qquad \Psi_j = \frac{s_j}{f}, \tag{3.1}$$

the mKPIESCS (2.14)–(2.16) can be transformed into the bilinear forms

$$D_x^2 g\cdot f - D_y g\cdot f = 0, \tag{3.2}$$

$$D_t g\cdot f + D_x^3 g\cdot f + 3D_xD_y g\cdot f = \sum_{j=1}^{N} h_js_j, \tag{3.3}$$

$$D_y h_j\cdot f - D_x^2 h_j\cdot f = 0, \tag{3.4}$$

$$D_y s_j\cdot g + D_x^2 s_j\cdot g = 0, \tag{3.5}$$

where D is the well-known Hirota bilinear operator

$$D_x^l D_y^m D_t^n a\cdot b = (\partial_x - \partial_{x'})^l(\partial_y - \partial_{y'})^m(\partial_t - \partial_{t'})^n a(x, y, t)b(x', y', t')|_{x'=x,y'=y,t'=t}.$$

Expanding f, g and h_j, s_j as the series

$$f = 1 + f^{(2)}\epsilon^2 + f^{(4)}\epsilon^4 + f^{(6)}\epsilon^6 + \cdots, \tag{3.6}$$

$$g = 1 + g^{(2)}\epsilon^2 + g^{(4)}\epsilon^4 + \cdots, \tag{3.7}$$

$$h_j = h_j^{(1)}\epsilon + h_j^{(3)}\epsilon^3 + \cdots, \tag{3.8}$$

$$s_j = s_j^{(1)}\epsilon + s_j^{(3)}\epsilon^3 + \cdots. \tag{3.9}$$

If we take

$$h_j^{(1)} = -\sqrt{2(k_j + q_j)\beta_j(t)}\, \mathrm{e}^{\xi_j}, \qquad \xi_j = k_j x + k_j^2 y - 4k_j^3 t - \int_0^t \beta_j(z)\,\mathrm{d}z + \xi_j^{(0)}, \tag{3.10}$$

$$s_j^{(1)} = \sqrt{2(k_j + q_j)\beta_j(t)}\, \mathrm{e}^{\eta_j}, \qquad \eta_j = q_j x - q_j^2 y - 4q_j^3 t - \int_0^t \beta_j(z)\,\mathrm{d}z + \eta_j^{(0)},$$
$$j = 1, 2 \ldots N, \tag{3.11}$$

where $k_j, q_j, \xi_j^{(0)}, \eta_j^{(0)}$ are all real constants and $\beta_j(t)$ is an arbitrary real function of t. Then through the standard process of Hirota method, it is easy to find the one-soliton and two-soliton solutions which can be presented respectively by

$$u = \left[\ln \frac{1 + a_1\,\mathrm{e}^{\xi_1+\eta_1}}{1 + b_1\,\mathrm{e}^{\xi_1+\eta_1}}\right]_x, \tag{3.12}$$

$$\Phi_1 = \frac{-\sqrt{2(k_1 + q_1)\beta_1(t)}\,\mathrm{e}^{\xi_1}}{1 + a_1\,\mathrm{e}^{\xi_1+\eta_1}}, \qquad \Psi_1 = \frac{\sqrt{2(k_1 + q_1)\beta_1(t)}\,\mathrm{e}^{\eta_1}}{1 + b_1\,\mathrm{e}^{\xi_1+\eta_1}}, \tag{3.13}$$

as shown in figure 1, and

$$u = \left[\frac{1 + a_1\,\mathrm{e}^{\xi_1+\eta_1} + a_2\,\mathrm{e}^{\xi_2+\eta_2} + a_1a_2\,\mathrm{e}^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}}{1 + b_1\,\mathrm{e}^{\xi_1+\eta_1} + b_2\,\mathrm{e}^{\xi_2+\eta_2} + b_1b_2\,\mathrm{e}^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}}\right]_x, \tag{3.14}$$

$$\Phi_1 = -\sqrt{2(k_1 + q_1)\beta_1(t)}\,\mathrm{e}^{\xi_1}\,\frac{1 + b_2\frac{(k_2-k_1)}{(k_1+q_2)}\,\mathrm{e}^{\xi_2+\eta_2+\mathrm{i}\pi}}{1 + a_1\,\mathrm{e}^{\xi_1+\eta_1} + a_2\,\mathrm{e}^{\xi_2+\eta_2} + a_1a_2\,\mathrm{e}^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}}, \tag{3.15}$$

$$\Phi_2 = -\sqrt{2(k_2 + q_2)\beta_2(t)}\,\mathrm{e}^{\xi_2}\,\frac{1 + b_1\frac{(k_2-k_1)}{(k_2+q_1)}\,\mathrm{e}^{\xi_1+\eta_1}}{1 + a_1\,\mathrm{e}^{\xi_1+\eta_1} + a_2\,\mathrm{e}^{\xi_2+\eta_2} + a_1a_2\,\mathrm{e}^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}}, \tag{3.16}$$

$$\Psi_1 = \sqrt{2(k_1 + q_1)\beta_1(t)}\,\mathrm{e}^{\eta_1}\,\frac{1 + a_2\frac{(q_2-q_1)}{(q_1+k_2)}\,\mathrm{e}^{\xi_2+\eta_2+\mathrm{i}\pi}}{1 + b_1\,\mathrm{e}^{\xi_1+\eta_1} + b_2\,\mathrm{e}^{\xi_2+\eta_2} + b_1b_2\,\mathrm{e}^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}}, \tag{3.17}$$

$$\Psi_2 = \sqrt{2(k_2 + q_2)\beta_2(t)}\,\mathrm{e}^{\eta_2}\,\frac{a_1\frac{(q_2-q_1)}{(q_2+k_1)}\,\mathrm{e}^{\xi_1+\eta_1}}{1 + b_1\,\mathrm{e}^{\xi_1+\eta_1} + b_2\,\mathrm{e}^{\xi_2+\eta_2} + b_1b_2\,\mathrm{e}^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}}. \tag{3.18}$$

Generally, we can conjecture the N-soliton solutions as

$$g = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j(\xi_j + \eta_j + \alpha_j) + \sum_{1\leqslant j<l}^{N} \mu_j\mu_l A_{jl}\right], \tag{3.19}$$

$$f = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j(\xi_j + \eta_j + \gamma_j) + \sum_{1\leqslant j<l}^{N} \mu_j\mu_l A_{jl}\right], \tag{3.20}$$

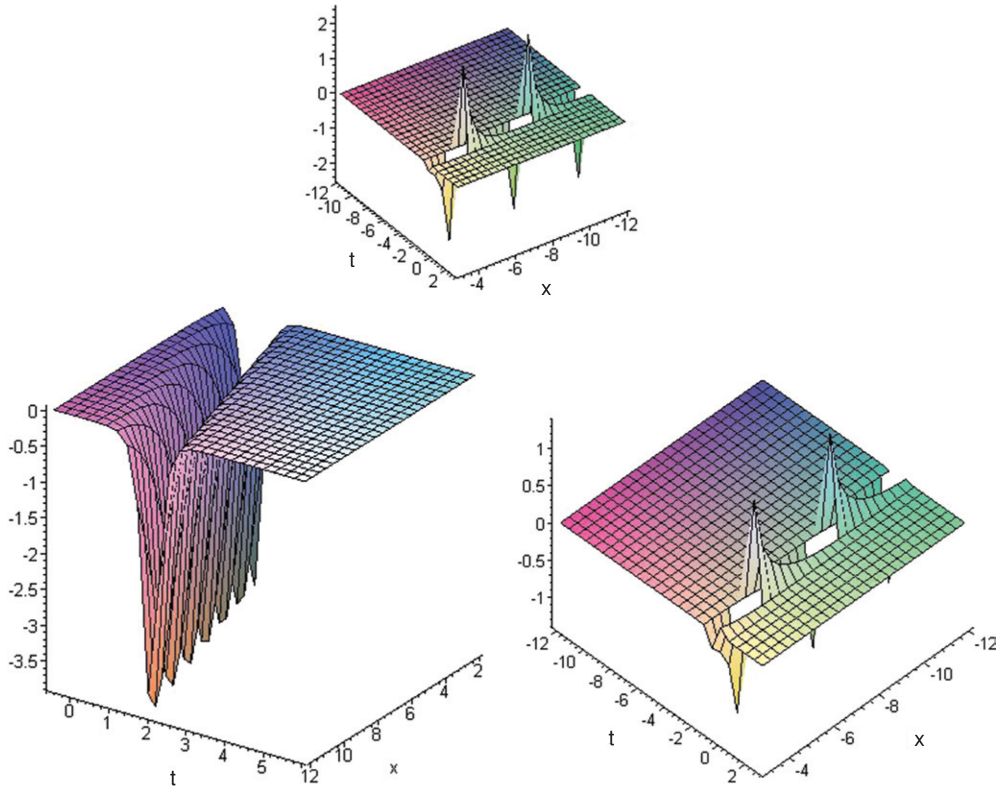

Figure 1. The one soliton solution (3.12) and (3.13) with $k_1 = 1, q_1 = 1, \beta_1 = 2, y = 1$.

$$h_m = -2\sqrt{2(k_m + q_m)\beta_m(t)}\, \mathrm{e}^{\xi_m} \sum_{\mu=0,1} \exp\left[\sum_{1\leqslant j<m} \mu_j(\xi_j + \eta_j + \gamma_j + B_{mj})\right] \times \exp\left[\sum_{j>m}^{N} \mu_j(\xi_j + \eta_j + \gamma_j + \mathrm{i}\pi + B_{jm}) + \sum_{1\leqslant j<l,j,l\neq m}^{N} \mu_j\mu_l A_{jl}\right], \qquad (3.21)$$

$$s_m = 2\sqrt{2(k_m + q_m)\beta_m(t)}\, \mathrm{e}^{\eta_m} \sum_{\mu=0,1} \exp\left[\sum_{1\leqslant j<m} \mu_j(\xi_j + \eta_j + \alpha_j + C_{mj})\right] \times \exp\left[\sum_{j>m}^{N} \mu_j(\xi_j + \eta_j + \alpha_j + \mathrm{i}\pi + C_{jm}) + \sum_{1\leqslant j<l,j,l\neq m}^{N} \mu_j\mu_l A_{jl}\right], \qquad (3.22)$$

$$\mathrm{e}^{A_{jl}} = \frac{(k_j - k_l)(q_j - q_l)}{(k_j + q_l)(k_l + q_j)}, \qquad \mathrm{e}^{B_{mj}} = \left(\frac{k_m - k_j}{k_m + q_j}\right), \qquad \mathrm{e}^{C_{mj}} = \left(\frac{q_m - q_j}{q_m + k_j}\right),$$
$$\mathrm{e}^{\alpha_j} = a_j = k_j, \qquad \mathrm{e}^{\gamma_j} = b_j = -q_j. \qquad (3.23)$$

Here the sum is taken over all possible combinations of $\mu_j = 0, 1$ $(j = 1, 2, \ldots, N)$, $k_j, q_j, \xi_j^{(0)}, \eta_j^{(0)}$ are all real constants and $\beta_j(t)$ is an arbitrary real function of t. When $\beta_j(t) = 0$, (3.19) and (3.20) is just the solution for mKPI equation (2.13) [26].

4. Solving the mKPIESCS by Wronskian method

4.1. Wronskian method for the mKPI equation

Through the transformation $u = \left(\ln \frac{g}{f}\right)_x$, the bilinear form of the mKPI equation is

$$D_x^2 g \cdot f - D_y g \cdot f = 0, \tag{4.1}$$

$$D_t g \cdot f + D_x^3 g \cdot f + 3D_x D_y g \cdot f = 0. \tag{4.2}$$

The mKPI equation has the Wronskian form solutions as follows:

$$f = \begin{vmatrix} \phi_1 & \partial\phi_1 & \cdots & \partial^{N-1}\phi_1 \\ \phi_2 & \partial\phi_2 & \cdots & \partial^{N-1}\phi_2 \\ \cdots & \cdots & \cdots & \cdots \\ \phi_N & \partial\phi_N & \cdots & \partial^{N-1}\phi_N \end{vmatrix} = |\phi, \phi^{(1)}, \ldots, \phi^{(N-1)}|$$

$$= |0, 1, \ldots, N-1| = |\widehat{N-1}|, \tag{4.3}$$

$$g = \begin{vmatrix} \partial\phi_1 & \partial^2\phi_1 & \cdots & \partial^N\phi_1 \\ \partial\phi_2 & \partial^2\phi_2 & \cdots & \partial^N\phi_2 \\ \cdots & \cdots & \cdots & \cdots \\ \partial\phi_N & \partial^2\phi_N & \cdots & \partial^N\phi_N \end{vmatrix} = |\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(N)}| = |1, 2, \ldots, N| = |\widetilde{N}|, \tag{4.4}$$

where ϕ_j satisfy

$$\phi_{j,y} = \phi_{j,xx}, \tag{4.5}$$

$$\phi_{j,t} = -4\phi_{j,xxx}. \tag{4.6}$$

From (4.5) and (4.6), it is easy to obtain

$$f_x = |\widehat{N-2}, N|, \qquad f_{xx} = |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \tag{4.7}$$

$$f_{xx} = |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|, \tag{4.8}$$

$$f_y = -|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \tag{4.9}$$

$$f_{xy} = -|\widehat{N-4}, N-2, N-1, N| + |\widehat{N-2}, N+2|, \tag{4.10}$$

$$f_t = -4[|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|], \tag{4.11}$$

$$g_x = |\widetilde{N-1}, N+1|, \; g_{xx} = |\widetilde{N-2}, N, N+1| + |\widetilde{N-1}, N+2|, \tag{4.12}$$

$$g_{xxx} = |\widetilde{N-3}, N-1, N, N+1| + 2|\widetilde{N-2}, n, N+2| + |\widetilde{N-1}, N+3|, \tag{4.13}$$

$$g_y = -|\widetilde{N-2}, N, N-1| + |\widetilde{N-1}, N+2|, \tag{4.14}$$

$$g_{xy} = -|\widetilde{N-3}, N-1, N, N+1| + |\widetilde{N-1}, N+3|, \tag{4.15}$$

$$g_t = -4[|\widetilde{N-3}, N-1, N, N+1| - |\widetilde{N-2}, N, N+2| + |\widetilde{N-1}, N+3|. \tag{4.16}$$

Substituting (4.7)–(4.16) into (4.1)–(4.2), we have

$$D_x^2 g \cdot f - D_y g \cdot f = g_{xx} f - 2g_x f_x + g f_{xx} - (g_y f - g f_y)$$
$$= 2|\widetilde{N-2}, N, N+1||0, \widetilde{N-2}, N-1| - 2|\widetilde{N-2}, N-1, N+1||0, \widetilde{N-2}, N|$$
$$+ 2|\widetilde{N-2}, N-1, N||0, \widetilde{N-2}, N+1| = 0. \tag{4.17}$$

$$
\begin{aligned}
&D_t g\cdot f+D_x^3 g\cdot f+3D_xD_y g\cdot f\\
&\quad= g_t f-gf_t+g_{xxx}f-3g_{xx}f_x+3g_xf_{xx}-gf_{xxx}+3(g_{xy}f-g_xf_y-g_yf_x+f_{xy}g)\\
&\quad= 6[-|\widetilde{N-3},N-1,N,N+1||0,\widetilde{N-3},N-2,N-1|\\
&\qquad-|\widetilde{N-3},N-2,N-1,N||0,\widetilde{N-3},N-1,N+1|\\
&\qquad+|\widetilde{N-3},N-2,N-1,N+1||0,\widetilde{N-3},N-1,N|]\\
&\qquad+6[|\widetilde{N-2},N,N+2||0,\widetilde{N-2},N-1|+|\widetilde{N-2},N-1,N||0,\widetilde{N-2},N+2|\\
&\qquad-|\widetilde{N-2},N-1,N+2||0,\widetilde{N-2},N|]=0.
\end{aligned}
\tag{4.18}
$$

4.2. Wronskian method for the mKPIESCS

In this section, we will derive the solution in the Wronskian form for mKPIESCS similarly as in [20].

The Wronskian form solutions for the mKPIESCS can be written as (4.3), (4.4) and

$$
h_m=-\sqrt{2(k_m+q_m)\beta_m(t)}\,\mathrm{e}^{\xi_m-\eta_m}\begin{vmatrix}
\psi_1 & \partial\psi_1 & \cdots & \partial^{N-2}\psi_1 & 0\\
\psi_2 & \partial\psi_2 & \cdots & \partial^{N-2}\psi_2 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
\psi_{m-1} & \partial\psi_{m-1} & \cdots & \partial^{N-2}\psi_{m-1} & 0\\
\psi_m & \partial\psi_m & \cdots & \partial^{N-2}\psi_m & 1\\
\psi_{m+1} & \partial\psi_{m+1} & \cdots & \partial^{N-2}\psi_{m+1} & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
\psi_N & \partial\psi_N & \cdots & \partial^{N-2}\psi_N & 0
\end{vmatrix},
\tag{4.19}
$$

$$
s_m=\sqrt{2(k_m+q_m)\beta_m(t)}\begin{vmatrix}
\partial\phi_1 & \partial^2\phi_1 & \cdots & \partial^{N-1}\phi_1 & 0\\
\partial\phi_2 & \partial^2\phi_2 & \cdots & \partial^{N-1}\phi_2 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
\partial\phi_{m-1} & \partial^2\phi_{m-1} & \cdots & \partial^{N-1}\phi_{m-1} & 0\\
\partial\phi_m & \partial^2\phi_m & \cdots & \partial^{N-1}\phi_m & 1\\
\partial\phi_{m+1} & \partial^2\phi_{m+1} & \cdots & \partial^{N-1}\phi_{m+1} & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
\partial\phi_N & \partial^2\phi_N & \cdots & \partial^{N-1}\phi_N & 0
\end{vmatrix},
\tag{4.20}
$$

where

$$\phi_j=\mathrm{e}^{\xi_j}+(-1)^{j-1}\,\mathrm{e}^{-\eta_j},\tag{4.21}$$

$$\psi_j=(k_m-k_j)(k_j+q_m)\,\mathrm{e}^{\xi_j}+(-1)^{j-1}(q_m-q_j)(q_j+k_m)\,\mathrm{e}^{-\eta_j},\qquad (j<m),\tag{4.22}$$

$$\psi_j=(k_j-k_m)(k_j+q_m)\,\mathrm{e}^{\xi_j}+(-1)^{j-1}(q_j-q_m)(q_j+k_m)\,\mathrm{e}^{-\eta_j},\qquad (j>m).\tag{4.23}$$

First, we show that the Wronskian determinants f, g and h_m, s_m satisfy the bilinear equation (3.3). Expanding f, g and f_t, g_t by the mth row, we have

$$f=\sum_{j=1}^{N}(-1)^{m+j}\partial^{j-1}(\mathrm{e}^{\xi_m}+(-1)^{m-1}\,\mathrm{e}^{-\eta_m})A_{mj},\tag{4.24}$$

$$f_t = \sum_{j=1}^{N}(-1)^{m+j}\partial^{j-1}\big[\big(-4k_j^3 - \beta_j(t)\big)\,\mathrm{e}^{\xi_m} + (-1)^{m-1}\big(4q_j^3 + \beta_j(t)\big)\,\mathrm{e}^{-\eta_m}\big]A_{mj}, \qquad (4.25)$$

$$g = \sum_{l=1}^{N}(-1)^{m+l}\partial^{l}[\mathrm{e}^{\xi_m} + (-1)^{m-1}\,\mathrm{e}^{-\eta_m}]C_{ml}, \qquad (4.26)$$

$$g_t = \sum_{l=1}^{N}(-1)^{m+l}\partial^{l}\big[\big(-4k_l^3 - \beta_l(t)\big)\,\mathrm{e}^{\xi_m} + (-1)^{m-1}\big(4q_l^3 + \beta_l(t)\big)\,\mathrm{e}^{-\eta_m}\big]C_{ml}, \qquad (4.27)$$

where A_{mj} and C_{ml} are the cofactors of f and g respectively. Obviously $C_{mN} = A_{m1}$.

In section 4.1, we have shown that f, g with $\beta_j(t) = 0$, $(j = 1, 2, \ldots, N)$ satisfies the bilinear equation of mKPI equation (4.2). So, what we should do is to prove that the sum of all coefficients for a fixed $\beta_j(t)$ on the two sides of (3.3) is equal.

Without loss of generality, the following discussion will be restricted to the case of $\beta_m(t)$. Because there is only the first term $D_t g\cdot f = g_t f - g f_t$ including $\beta_m(t)$ and note the equality

$$\begin{aligned}&[\partial^l(\mathrm{e}^{\xi_m} - (-1)^{m-1}\,\mathrm{e}^{-\eta_m})][\partial^{j-1}(\mathrm{e}^{\xi_m} + (-1)^{m-1}\,\mathrm{e}^{-\eta_m})]\\ &\qquad - [\partial^l(\mathrm{e}^{\xi_m} + (-1)^{m-1}\,\mathrm{e}^{-\eta_m})][\partial^{j-1}(\mathrm{e}^{\xi_m} - (-1)^{m-1}\,\mathrm{e}^{-\eta_m})]\\ &\quad = 2(-1)^{m-1}\,\mathrm{e}^{\xi_m-\eta_m}\big[k_m^l(-q_m)^{j-1} - k_m^{j-1}(-q_m)^l\big],\end{aligned} \qquad (4.28)$$

then the term for $\beta_m(t)$ on the left-hand side of (3.3) can be written as

$$\begin{aligned}-2\beta_m(t)(-1)^{m-1}\,\mathrm{e}^{\xi_m-\eta_m}\Bigg\{&\sum_{j=1}^{N-1}\sum_{l=1}^{N-1}(-1)^{l+j}\big[k_m^l(-q_m)^{j-1} - k_m^{j-1}(-q_m)^l\big]C_{ml}A_{mj}\\ &+\sum_{l=1}^{N-1}(-1)^{l+N}\big[k_m^l(-q_m)^{N-1} - k_m^{N-1}(-q_m)^l\big]C_{ml}A_{mN}\\ &+\sum_{j=1}^{N-1}(-1)^{j+N}\big[k_m^N(-q_m)^{j-1} - k_m^{j-1}(-q_m)^N\big]C_{mN}A_{mj}\\ &+\big[k_m^N(-q_m)^{N-1} - k_m^{N-1}(-q_m)^N\big]C_{mN}A_{mN}\Bigg\}.\end{aligned} \qquad (4.29)$$

By means of the general determinant identity [20]

$$|Q, a, b||Q, c, d| - |Q, a, c||Q, b, d| + |Q, a, d||Q, b, c| = 0, \qquad (4.30)$$

where Q is an $(N-1)\times(N-3)$ matrix and a, b, c and d represent $N-1$ column vectors, it is not difficult to prove that

$$C_{mj} = |M(j), N|, \qquad j = 1, 2, \ldots, N-1, \qquad (4.31)$$

$$A_{mj} = |0, M(j)|, \qquad j = 1, 2, \ldots, N-1, \qquad (4.32)$$

$$C_{mj}A_{m,l+1} - C_{ml}A_{m,j+1} = |0, M(l, j), N|C_{mN}, \qquad (1 \leqslant l < j \leqslant N-3), \qquad (4.33)$$

$$C_{m,N-1}A_{m,j+1} - C_{mj}A_{mN} = |0, M(j, N-1), N|C_{mN}, \qquad j = 1, 2, \ldots, N-2, \qquad (4.34)$$

where the matrix $M(l, j)$ is defined by

$$M(l, j) = |1, 2, \ldots, l-1, l+1, \ldots, j-1, j+1, \ldots, N-1|_{(N-1)\times(N-3)}, \tag{4.35}$$

$$M(j) = |1, 2, \ldots, j-1, j+1, \ldots, N-1|_{(N-1)\times(N-2)}. \tag{4.36}$$

Using (4.31)–(4.36), expression (4.29) becomes

$$\begin{aligned} -2\beta_m(t)(-1)^{m-1}\,\mathrm{e}^{\xi_m-\eta_m}\Bigg\{ &\sum_{l=1}^{N-3}\sum_{j=l+1}^{N-2}(-k_mq_m)^l\big[q_m^{j-l}-(-k_m)^{j-l}\big]|0, M(l, j), N| \\ &+\sum_{j=1}^{N-2}(-k_mq_m)^j\big[q_m^{N-1-j}-(-k_m)^{N-1-j}\big]|0, M(j, N-1), N| \\ &+\sum_{l=1}^{N-1}\big[q_m^l-(-k_m)^l\big]C_{ml}+\sum_{j=1}^{N-1}(-k_mq_m)^{j-1}\big[q_m^{N+1-j}-(-k_m)^{N+1-j}\big]A_{mj} \\ &+(-k_mq_m)^{N-1}(k_m+q_m)A_{mN}\Bigg\}C_{mN}. \end{aligned} \tag{4.37}$$

Now we turn about h_m and s_m. Obviously, from (4.20) we have

$$s_m = \sqrt{2(k_m+q_m)\beta_m(t)}(-1)^{m+N}C_{mN}. \tag{4.38}$$

While from (4.19), h_m can be written as

$$h_m = -\sqrt{2(k_m+q_m)\beta_m(t)}\,\mathrm{e}^{\xi_m-\eta_m}\tilde{h}_m, \tag{4.39}$$

where $\tilde{h}_m$ is an $N\times N$ determinant

$$\begin{aligned} \tilde{h}_m &= \begin{vmatrix} -L\phi_1 & -L\phi_1^{(1)} & \cdots & -L\phi_1^{(N-2)} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -L\phi_{m-1} & -L\phi_m^{(1)} & \cdots & -L\phi_m^{(N-2)} & 0 \\ L\phi_m & L\phi_m^{(1)} & \cdots & L\phi_m^{(N-2)} & 1 \\ L\phi_{m+1} & L\phi_{m+1}^{(1)} & \cdots & L\phi_{m+1}^{(N-2)} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L\phi_N & L\phi_N^{(1)} & \cdots & L\phi_N^{(N-2)} & 0 \end{vmatrix} \\ &= (-1)^{m-1}|L\cdot 0, L\cdot 1, L\cdot 2, \ldots, L\cdot(N-2), \tau_m| \\ &= |b\cdot 0+a\cdot 1+2, b\cdot 1+a\cdot 2+3, b\cdot 2+a\cdot 3+4, \ldots, b\cdot(N-2) \\ &\quad +a\cdot(N-1)+N, \tau_m| \end{aligned} \tag{4.40}$$

$L = b + a\partial + \partial^2, \quad b = -k_mq_m, \quad a = q_m - k_m, \quad \tau_m = (\delta_{m,1}, \delta_{m,2}, \ldots, \delta_{m,N})^T.$

By simple analysis for $N\times N$ determinant

$$|F(l, j), \tau_m| = |0, 1, \ldots, l-1, l+1, \ldots, j-1, j+1, j+2, \ldots, N, \tau_m|, \tag{4.41}$$

we obtain

$$\begin{aligned} \tilde{h}_m = (-1)^{m-1}\sum_{l=0}^{N-1}\sum_{j=l+1}^{N} &|F(l, j), \tau_m|(-k_mq_m)^l\big[(q_m-k_m)^{j-l-1} \\ &- C_{j-l-2}^1(q_m-k_m)^{j-l-3}(-k_mq_m)+C_{j-l-3}^2(q_m-k_m)^{j-l-5}(-k_mq_m)^2 \\ &- C_{j-l-4}^3(q_m-k_m)^{j-l-7}(-k_mq_m)^3+\cdots \\ &+\begin{cases} (-1)^{\frac{j-l-1}{2}}(-k_mq_m)^{\frac{j-l-1}{2}}\big] & \text{if} \quad j-l \text{ is odd} \\ (-1)^{\frac{j-l-2}{2}}C_{\frac{j-l}{2}}^{\frac{j-l-2}{2}}(-k_mq_m)^{\frac{j-l-2}{2}}(q_m-k_m)\Big] & \text{if} \quad j-l \text{ is even.} \end{cases} \end{aligned} \tag{4.42}$$

We show further the algebraic sum after $|F(l,\, j),\, \tau_m|$ be expressed as [20]

$$(-k_m q_m)^l \frac{q_m^{j-l} - (-k_m)^{j-l}}{q_m + k_m}. \tag{4.43}$$

As a result, we obtain immediately

$$\begin{aligned}
h_m = {} & (-1)^m \sqrt{2(k_m + q_m)\beta_m(t)}\, \mathrm{e}^{\xi_m - \eta_m} \\
& \times \left\{ \sum_{l=1}^{N-3} \sum_{j=l+1}^{N-2} (-k_m q_m)^l \left(\frac{q_m^{j-l} - (-k_m)^{j-l}}{q_m + k_m} \right) |P(l,\, j),\, N-1,\, N,\, \tau_m| \right. \\
& + \sum_{j=1}^{N-1} \left(\frac{q_m^{j} - (-k_m)^{j}}{q_m + k_m} \right) |P(0,\, j),\, N-1,\, N,\, \tau_m| \\
& + \sum_{l=1}^{N-2} (-k_m q_m)^l \left(\frac{q_m^{N-1-l} - (-k_m)^{N-1-l}}{q_m + k_m} \right) |P(l),\, N,\, \tau_m| \\
& + \sum_{l=0}^{N-2} (-k_m q_m)^l \left(\frac{q_m^{N-l} - (-k_m)^{N-l}}{q_m + k_m} \right) |P(l),\, N-1,\, \tau_m| \\
& \left. + (-k_m q_m)^{N-1} |\widehat{N-2},\, \tau_m| \right\}.
\end{aligned} \tag{4.44}$$

Here $P(l,\, j)$ is the $N \times (N-3)$ matrix without l column and j column, $P(l)$ is the $N \times (N-2)$ matrix without l column. Obviously we have

$$\begin{aligned}
& |P(l,\, j),\, N-1,\, N,\, \tau_m| = |0,\, M(l,\, j),\, N|(-1)^{m+N},\quad |P(l),\, N,\, \tau_m| \\
& \qquad = |0,\, M(l,\, N-1),\, N|(-1)^{m+N}, \\
& |P(l),\, N-1,\, \tau_m| = |0,\, M(l)|(-1)^{m+N}, \qquad P|(0,\, j),\, N-1,\, N,\, \tau_m| = C_{mj}(-1)^{m+N}, \\
& |N-2,\, \tau_m| = A_{mN}(-1)^{m+N}.
\end{aligned} \tag{4.45}$$

So

$$\begin{aligned}
h_m s_m = {} & (-1)^m 2\beta_m(t)\, \mathrm{e}^{\xi_m - \eta_m} \left\{ \sum_{l=1}^{N-3} \sum_{j=l+1}^{N-2} (-k_m q_m)^l \big[q_m^{j-l} - (-k_m)^{j-l}\big] |0,\, M(l,\, j),\, N| \right. \\
& + \sum_{j=1}^{N-1} \big[q_m^{j} - (-k_m)^{j}\big] C_{mj} \\
& + \sum_{l=1}^{N-2} (-k_m q_m)^l \big[q_m^{N-1-l} - (-k_m)^{N-1-l}\big] |0,\, M(l,\, N-1),\, N| \\
& + \sum_{l=0}^{N-2} (-k_m q_m)^l \big[q_m^{N-l} - (-k_m)^{N-l}\big] A_{ml} \\
& \left. + (-k_m q_m)^{N-1} (k_m + q_m) A_{mN} \right\} C_{mN},
\end{aligned} \tag{4.46}$$

therefore the term for $\beta_m(t)$ on the right-hand side of (3.3) equals (4.37). That is to say that Wronskian form (4.3), (4.4) and (4.19), (4.20) satisfy equation (3.3).

Next, we will prove that h_m and f satisfy (3.4). Using the abbreviated notation, it can be obtained

$$h_{m,y} = (-1)^m\sqrt{2(k_m+q_m)\beta_m(t)}\,\mathrm{e}^{\xi_m-\eta_m}\big[-|L(\widehat{N-4}), L(N-2), L(N-1), \tau_m| + |L(\widehat{N-3}), LN, \tau_m| + \big(k_m^2+q_m^2\big)|L(\widehat{N-2}), \tau_m|\big] \tag{4.47}$$

$$h_{m,x} = (-1)^m\sqrt{2(k_m+q_m)\beta_m(t)}\,\mathrm{e}^{\xi_m-\eta_m}[|L(\widehat{N-3}), L(N-1), \tau_m| + (k_m-q_m)|L(\widehat{N-2}), \tau_m|], \tag{4.48}$$

$$h_{m,xx} = (-1)^m\sqrt{2(k_m+q_m)\beta_m(t)}\,\mathrm{e}^{\xi_m-\eta_m}[|L(\widehat{N-4}), L(N-2), L(N-1), \tau_m| + |L(\widehat{N-3}), LN, \tau_m| + 2(k_m-q_m)|L(\widehat{N-3}), L(N-1), \tau_m| + (k_m-q_m)^2|L(\widehat{N-2}), \tau_m|]. \tag{4.49}$$

Substituting (4.47)–(4.49) and Wronskian f and its related derivatives (4.7) into bilinear equation (3.4) gives

$$[-|L(\widehat{N-4}), L(N-2), L(N-1), \tau_m| + (q_m-k_m)|L(\widehat{N-3}), L(N-1), \tau_m| + k_mq_m|L(\widehat{N-2}), \tau_m|]|\widehat{N-1}| + [|L(\widehat{N-3}), L(N-1), \tau_m| - (q_m-k_m)|L(\widehat{N-2}), \tau_m|]|\widehat{N-2}, N| - |L(\widehat{N-2}), \tau_m||\widehat{N-2}, N+1| = 0. \tag{4.50}$$

But we can work out that

$$\begin{aligned}(q_m+k_m)&|L(\widehat{N-4}), L(N-2), L(N-1), \tau_m| \\ &= \sum_{l=0}^{N-3}\sum_{j=l+1}^{N-2}(-k_mq_m)^l[q_m^{j-l}(-k_m)^{j-l}][(q_m-k_m)^2+k_mq_m]|P(l,j), N-1, N, \tau_m| \\ &+ \sum_{l=0}^{N-3}\sum_{j=l+1}^{N-2}(-k_mq_m)^l\big[q_m^{j-l}-(-k_m)^{j-l}\big](q_m-k_m)|P(l,j), N-1, N+1, \tau_m| \\ &+ \sum_{l=0}^{N-3}\sum_{j=l+1}^{N-2}(-k_mq_m)^l\big[q_m^{j-l}-(-k_m)^{j-l}\big]|P(l,j), N, N+1, \tau_m| \\ &+ \sum_{l=0}^{N-2}(-k_mq_m)^{l+2}\big[q_m^{N-2-l}-(-k_m)^{N-2-l}\big]|P(l), N-1, \tau_m| \\ &+ \sum_{l=0}^{N-2}(-k_mq_m)^{l+1}\big[q_m^{N-2-l}-(-k_m)^{N-2-l}\big](q_m-k_m)|P(l), N, \tau_m| \\ &+ \sum_{l=0}^{N-2}(-k_mq_m)^{l+1}\big[q_m^{N-2-l}-(-k_m)^{N-2-l}\big]|P(l), N+1, \tau_m|,\end{aligned} \tag{4.51}$$

$$\begin{aligned}(q_m+k_m)&|L(\widehat{N-3}), L(N-1), \tau_m| \\ &= \sum_{l=0}^{N-3}\sum_{j=l+1}^{N-2}(-k_mq_m)^l\big[q_m^{j-l}-(-k_m)^{j-l}\big](q_m-k_m)|P(l,j), N-1, N, \tau_m|\end{aligned}$$

$$+\sum_{l=0}^{N-3}\sum_{j=l+1}^{N-2}(-k_mq_m)^l\big[q_m^{j-l}-(-k_m)^{j-l}\big]|P(l,j),N-1,N+1,\tau_m|$$
$$+\sum_{l=0}^{N-2}(-k_mq_m)^{l+1}\big[q_m^{N-1-l}-(-k_m)^{N-1-l}\big]|P(l),N-1,\tau_m|$$
$$+\sum_{l=0}^{N-2}(-k_mq_m)^l\big[q_m^{N-1-l}-(-k_m)^{N-1-l}\big](q_m-k_m)|P(l),N,\tau_m|$$
$$+\sum_{l=0}^{N-2}(-k_mq_m)^l\big[q_m^{N-1-l}-(-k_m)^{N-1-l}\big]|P(l),N+1,\tau_m|. \tag{4.52}$$

Inserting (4.44), (4.51) and (4.52) into the left-hand side of (4.50) leaves only the terms

$$\begin{aligned}
&\sum_{l=0}^{N-3}\sum_{j=l+1}^{N-2}(-k_mq_m)^l\big[q_m^{j-l}-(-k_m)^{j-l}\big](-|P(l,j),N-1,N,\tau_m||\widehat{N-2},N+1|\\
&\quad+|P(l,j),N-1,N+1,\tau_m||\widehat{N-2},N|-|P(l,j),N,N+1,\tau_m||\widehat{N-1}|)\\
&\quad+\sum_{l=0}^{N-2}(-k_mq_m)^l\big[q_m^{N+1-l}-(-k_m)^{N+1-l}\big](|P(l),N,\tau_m||\widehat{N-1}|\\
&\quad-|P(l),N-1,\tau_m||\widehat{N-2},N|)\\
&\quad+\sum_{l=0}^{N-2}(-k_mq_m)^l\big[q_m^{N-l}-(-k_m)^{N-l}\big](|P(l),N+1,\tau_m||\widehat{N-1}|\\
&\quad-|P(l),N-1,\tau_m||\widehat{N-2},N+1|)\\
&\quad+\sum_{l=0}^{N-2}(-k_mq_m)^l\big[q_m^{N-1-l}-(-k_m)^{N-1-l}\big](|P(j),N+1,\tau_m||\widehat{N-2},N|\\
&\quad-|P(m),N,\tau_m||\widehat{N-2},N+1|)+(-k_mq_m)^{N-1}(q_m+k_m)|\widehat{N-2},\tau_m|\\
&\quad\times[k_mq_m|\widehat{N-1}|-(q_m-k_m)|\widehat{N-2},N|-|\widehat{N-2},N+1|],
\end{aligned} \tag{4.53}$$

noting that

$$\begin{aligned}
&-|P(l,j),N-1,N,\tau_m||\widehat{N-2},N+1|+|P(l,j),N-1,N+1,\tau_m||\widehat{N-2},N|\\
&\quad-|P(l,j),N,N+1,\tau_m||\widehat{N-1}|\\
&\quad=(-1)^{N+1}\begin{vmatrix}P(l,j) & O & 0 & 0 & N-1 & N & N+1 & \tau_m\\ O & P(l,j) & l & j & N-1 & N & N+1 & \tau_m\end{vmatrix}\\
&\quad-|P(l,j),N-1,N,N+1||\widehat{N-2},\tau_m|\\
&\quad=-|P(l,j),N-1,N,N+1||\widehat{N-2},\tau_m|,
\end{aligned} \tag{4.54}$$

$$\begin{aligned}
&|P(l),N,\tau_m||\widehat{N-1}|-|P(l),N-1,\tau_m||\widehat{N-2},N|\\
&\quad=\begin{vmatrix}P(l) & O & 0 & N-1 & N & \tau_m\\ O & P(l) & l & N-1 & N & \tau_m\end{vmatrix}-|P(l),N-1,N||\widehat{N-2},\tau_m|\\
&\quad=-|P(l),N-1,N||\widehat{N-2},\tau_m|,
\end{aligned} \tag{4.55}$$

$$
\begin{aligned}
&[|P(l), N+1, \tau_m||\widehat{N-1}| - |P(l), N-1, \tau_m||\widehat{N-2}, N+1|] \\
&\quad = \begin{vmatrix} P(l) & O & 0 & N-1 & N+1 & \tau_m \\ O & P(l) & l & N-1 & N+1 & \tau_m \end{vmatrix} - |P(l), N-1, N+1||\widehat{N-2}, \tau_m| \\
&\quad = -|P(l), N-1, N+1||\widehat{N-2}, \tau_m|,
\end{aligned} \tag{4.56}
$$

$$
\begin{aligned}
&[|P(l), N+1, \tau_m||\widehat{N-1}| - |P(l), N-1, \tau_m||\widehat{N-2}, N+1|] \\
&\quad = \begin{vmatrix} P(l) & O & 0 & N & N+1 & \tau_m \\ O & P(l) & l & N & N+1 & \tau_m \end{vmatrix} - |P(l), N, N+1||\widehat{N-2}, \tau_m| \\
&\quad = -|P(l), N, N+1||\widehat{N-2}, \tau_m|,
\end{aligned} \tag{4.57}
$$

then (4.53) reduce to

$$
\begin{aligned}
(-|\widehat{N-2}, \tau_m|)\Bigg\{ &\sum_{l=0}^{N-3} \sum_{j=l+1}^{N-2} (-k_m q_m)^l \big[q_m^{j-l} - (-k_{m1})^{j-l}\big] |P(l, j), N-1, N, N+1| \\
&+ \sum_{l=0}^{N-2} (-k_m q_m)^l \big[q_m^{N+1-l} - (-k_m)^{N+1-l}\big] |P(l), N-1, N| \\
&+ \sum_{l=0}^{N-2} (-k_m q_m)^l \big[q_m^{N-l} - (-k_m)^{N-l}\big] |P(l), N-1, N+1| \\
&+ \sum_{l=0}^{N-2} (-k_m q_m)^l \big[q_m^{N-1-l} - (-k_m)^{N-1-l}\big] |P(l), N, N+1| \\
&- (-k_m q_m)^{N-1} (q_m + k_m) [k_m q_m |\widehat{N-1}| \\
&+ (k_m - q_m) |\widehat{N-2}, N| - |\widehat{N-2}, N+1|] \Bigg\},
\end{aligned} \tag{4.58}
$$

which is just $-|\widehat{N-2}, \tau_m||\widehat{L(N+1)}| = 0$.

At last, the remained problem is to prove that Wronskian s_m and g satisfy equation (3.5).

$$
\begin{aligned}
s_{m,x} &= \sqrt{2(k_m + q_m)\beta_m(t)} |\widetilde{N-2}, N, \tau_m|, \\
s_{m,xx} &= \sqrt{2(k_m + q_m)\beta_m(t)} [|\widetilde{N-3}, N-1, N, \tau_m| + |\widetilde{N-2}, N+1, \tau_m|],
\end{aligned} \tag{4.59}
$$

$$
s_{m,y} = \sqrt{2(k_m + q_m)\beta_m(t)} [-|\widetilde{N-3}, N-1, N, \tau_m| + |\widetilde{N-2}, N+1, \tau_m|]. \tag{4.60}
$$

Substituting (4.12)–(4.14) and (4.59)–(4.60) into (3.5)

$$
\begin{aligned}
&D_y s_m \cdot g + D_x^2 s_m \cdot g = s_{m,y} g - s_m g_y + s_{m,xx} g - 2 s_{m,x} g_x + s_m g_{xx} \\
&\quad = \sqrt{2(k_m + q_m)\beta_m(t)} [|\widetilde{N-2}, N+1, \tau_m||\widetilde{N-2}, N-1, N| \\
&|\widetilde{N-2}, N, \tau_m||\widetilde{N-2}, N-1, N+1| + |\widetilde{N-2}, N-1, \tau_m||\widetilde{N-2}, N, N+1|] = 0.
\end{aligned} \tag{4.61}
$$

5. Coincidence of these solutions

By now, we have found two kinds solutions of the bilinear equations (3.2)–(3.5), where (4.3)–(4.4) and (4.19)–(4.20) are just verified whereas (3.19)–(3.22) is only conjectured. In this section, we will show that these two kinds of solutions are the same for recovering the N-soliton solutions from the transformation (3.1).

By virtue of the addition rule of determinants, (4.3) can be represented by the sum of 2^{N-1} Vandermonde determinants. So we have

$$f = \sum_{\epsilon=0,1}(2\epsilon_2-1)(2\epsilon_4-1)\cdots\left(2\epsilon_{[\frac{N}{2}]}-1\right)\Delta(\epsilon_1k_1+(\epsilon_1-1)q_1,\epsilon_2k_2 \\ +(\epsilon_2-1)q_2,\ldots,\epsilon_Nk_N+(\epsilon_N-1)q_N)\exp\left\{\sum_{j=1}^{N}[\epsilon_j\xi_j+(\epsilon_j-1)\eta_j]\right\} \\ = (-1)^{\frac{N(N-1)}{2}}\sum_{\epsilon=0,1}\prod_{1\leqslant j<l}(2\epsilon_l-1)[\epsilon_jk_j+(\epsilon_j-1)q_j-\epsilon_lk_l-(\epsilon_l-1)q_l] \\ \times\exp\left\{\sum_{j=1}^{N}[\epsilon_j\xi_j+(\epsilon_j-1)\eta_j]\right\}, \tag{5.1}$$

where $\Delta(\epsilon_1k_1+(\epsilon_1-1)q_1,\epsilon_2k_2+(\epsilon_2-1)q_2,\ldots,\epsilon_Nk_N+(\epsilon_N-1)q_N)$ denotes an $N\times N$ Vandermonde determinant with the entries $\epsilon_1k_1+(\epsilon_1-1)q_1,\epsilon_2k_2+(\epsilon_2-1)q_2,\ldots,\epsilon_Nk_N+(\epsilon_N-1)q_N$ and the sum over $\epsilon=0,1$ refers to each of the $\epsilon_j=0,1$ $(j=1,2,\ldots,N)$.

Noting that

$$\frac{(2\epsilon_l-1)[\epsilon_jk_j+(\epsilon_j-1)q_j-\epsilon_lk_l-(\epsilon_l-1)q_l]}{q_j-q_l} \\ = \left(\frac{k_j+q_l}{q_l-q_j}\right)^{(1-\epsilon_l)\epsilon_j}\left(\frac{k_l+q_j}{q_l-q_j}\right)^{(1-\epsilon_j)\epsilon_l}\left(\frac{k_l-k_j}{q_l-q_j}\right)^{\epsilon_j\epsilon_l} \\ = \left(\frac{k_j+q_l}{q_l-q_j}\right)^{\epsilon_j}\left(\frac{k_l+q_j}{q_l-q_j}\right)^{\epsilon_l}\left[\frac{(k_l-k_j)(q_l-q_j)}{(k_j+q_l)(k_l+q_j)}\right]^{\epsilon_j\epsilon_l}, \tag{5.2}$$

and

$$\prod_{1\leqslant j<l\leqslant N}\left[\frac{(k_l-k_j)(q_l-q_j)}{(k_j+q_l)(k_l+q_j)}\right]^{\epsilon_j\epsilon_l} = \exp\left(\sum_{1\leqslant j<l}^{N}\epsilon_j\epsilon_lA_{jl}\right),$$

(5.1) becomes

$$f = \prod_{1\leqslant j<l}^{N}(q_l-q_j)\,\mathrm{e}^{\sum_{j=1}^{N}(-\eta_j)}\sum_{\epsilon=0,1}\exp\left[\sum_{j=1}^{N}\epsilon_j(\xi_j'+\eta_j')+\sum_{1\leqslant j<l}^{N}\epsilon_j\epsilon_lA_{jl}\right], \tag{5.3}$$

where

$$\mathrm{e}^{\xi_j}\prod_{j\neq l}(k_j+q_l) = \mathrm{e}^{\xi_j'}, \qquad \mathrm{e}^{\eta_j}\prod_{j>l}(q_j-q_l)^{-1}\prod_{l>j}(q_l-q_j)^{-1} = \mathrm{e}^{-\eta_j'}. \tag{5.4}$$

Similarly,

$$g = \prod_{1\leqslant j<l}^{N}(q_l-q_j)\,\mathrm{e}^{\sum_{j=1}^{N}(-\eta_j-b_j)}\sum_{\epsilon=0,1}\exp\left[\sum_{j=1}^{N}\epsilon_j(\xi_j'+a_j+\eta_j'+b_j)+\sum_{1\leqslant j<l}^{N}\epsilon_j\epsilon_lA_{jl}\right]. \tag{5.5}$$

It may be seen that the soliton solution constructed from (5.3) and (5.5) is the same as that from (3.19)–(3.22) with $\mathrm{e}^{\alpha_j}=-\frac{k_j}{q_j}$, $\mathrm{e}^{\gamma_j}=1$.

We can deal with s_m and h_m in a similar way to f, g. (4.20) can be rewritten as

$$s_m = \sqrt{2(k_m+q_m)\beta_m(t)} \cdot \begin{vmatrix} \partial\phi_1 & \partial^2\phi_1 & \cdots & \partial^{N-1}\phi_1 & 0 \\ \partial\phi_2 & \partial^2\phi_2 & \cdots & \partial^{N-1}\phi_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial\phi_{m-1} & \partial^2\phi_{m-1} & \cdots & \partial^{N-1}\phi_{m-1} & 0 \\ \partial\phi_m & \partial^2\phi_m & \cdots & \partial^{N-1}\phi_m & 1 \\ \partial\phi'_{m+1} & \partial^2\phi'_{m+1} & \cdots & \partial^{N-1}\phi'_{m+1} & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial\phi'_N & \partial^2\phi'_N & \cdots & \partial^{N-1}\phi'_N & 0 \end{vmatrix}, \tag{5.6}$$

where $\phi'_j = (-\mathrm{i})[\mathrm{e}^{\xi_j+\frac{\pi}{2}i} + (-1)^j\,\mathrm{e}^{-\eta_j-\frac{\pi}{2}\mathrm{i}}]$. So

$$s_m = \sqrt{2(k_m+q_m)\beta_m(t)} \prod_{1\leqslant j<l, j,l\neq m} (q_j - q_l) \exp\left[\sum_{j=1, j\neq m}^{N} (-\eta_j - b_j)\right] \sum_{\epsilon=0,1}$$
$$\times \exp\left[\sum_{j<m} \epsilon_j(\xi'_j + \eta'_j + C_{mj}) + \sum_{j>m} \epsilon_j(\xi'_j + a_j + \eta'_j + b_j + i\pi + C_{jm}) + \sum_{1\leqslant j<l}^{N} \epsilon_j\epsilon_l A_{jl}\right]. \tag{5.7}$$

From (4.19), we have

$$h_m = -\sqrt{2(k_m+q_m)\beta_m(t)}\,\mathrm{e}^{\xi_m-\eta_m} \begin{vmatrix} \psi_1 & \partial\psi_1 & \cdots & \partial^{N-2}\psi_1 & 0 \\ \psi_2 & \partial\psi_2 & \cdots & \partial^{N-2}\psi_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_{m-1} & \partial\psi_{m-1} & \cdots & \partial^{N-2}\psi_{m-1} & 0 \\ \psi_m & \partial\psi_m & \cdots & \partial^{N-2}\psi_m & 1 \\ \psi'_{m+1} & \partial\psi'_{m+1} & \cdots & \partial^{N-2}\psi'_{m+1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi'_N & \partial\psi'_N & \cdots & \partial^{N-2}\psi'_N & 0 \end{vmatrix}, \tag{5.8}$$

where

$$\psi'_j = (-\mathrm{i})[(k_j-k_m)(k_j+q_m)\,\mathrm{e}^{\xi_j+\frac{\pi}{2}\mathrm{i}} + (-1)^j(q_j-q_m)(q_j+k_m)\,\mathrm{e}^{-\eta_j-\frac{\pi}{2}i}], \qquad (j>m). \tag{5.9}$$

It is easy to derive

$$h_m = -\sqrt{2(k_m+q_m)\beta_m(t)}(-1)^{N-m}(-\mathrm{i})^{N-m}\sum_{\epsilon=0,1}\prod_{k=1}^{[\frac{N}{2}]}(2\epsilon_{2k}-1)\prod_{1\leqslant j<l}^{N}[\epsilon_l k_l + (\epsilon_l-1)q_l - \epsilon_j k_j$$
$$- (\epsilon_j-1)q_j]\exp\left\{\sum_{j=1}^{N}[\epsilon_j\xi_j + (\epsilon_j-1)\eta_j]\right\}\prod_{j<m}\left\{[(k_m-k_j)(k_j+q_m)]^{\epsilon_j}\right.$$
$$\left.\times[(q_m-q_j)(q_j+k_m)]^{1-\epsilon_j}\right\}$$
$$\times\prod_{j>m}\{[(k_j-k_m)(k_j+q_m)]^{\epsilon_j}[(q_j-q_m)(q_j+k_m)]^{1-\epsilon_j}\}$$
$$= -\sqrt{2(k_m+q_m)\beta_m}\prod_{1\leqslant j<l}(q_l-q_j)\prod_{j\neq m}(q_j+k_m)\,\mathrm{e}^{\xi_m}\exp\left[\sum_{j=1}^{N}(-\eta_j)\right]\sum_{\epsilon=0,1}$$

$$\times \exp\left[\sum_{j<m}\epsilon_j(\xi'_j+\eta'_j+B_{mj})+\sum_{j>m}\epsilon_j(\xi'_j+\eta'_j+i\pi+B_{jm})+\sum_{1\leqslant j<l}^{N}\epsilon_j\epsilon_l A_{jl}\right]. \tag{5.10}$$

From (5.3), (5.5) and (5.7), (5.10), we find that the solution Φ_j, Ψ_j obtained by the Wronskian form is identical with that obtained by the Hirota method with $\mathrm{e}^{\alpha_j} = -\frac{k_j}{q_j}$, $\mathrm{e}^{\gamma_j} = 1$.

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